Product-Mix Auctions

Paul Klemperer, 2018*

NEW Appendices giving more detail about the bidding languages,† and about Iceland's implementation, were added in 2018. The first section was added in 2015. The remainder is almost unchanged from the Discussion Paper, Klemperer (2008), that was published as Klemperer (2010).

*(pro bono) Software** to run Product-Mix Auctions is available here*

The "Product-Mix Auction" is a single-round auction that can be used whenever an auctioneer wants to sell (or buy) multiple differentiated goods. It allows all participants to express their preferences *between* varieties, as well as for alternative quantities of specific varieties. Bidders simultaneously make sets of bids. Each of a bidder's bids can be an "OR" bid that offers a different price for each different variety, and the auction then accepts at most one of the offers from each bid (whichever is best for the bidder given the prices that the auction sets). The graphical solution method makes the operation and merits of the auction easy to explain. The auction is more efficient, and less sensitive to market power, than *either* running a separate auction for each different variety *or* running a combined auction with predetermined price differences between varieties. It is also faster, less vulnerable to collusion, and can be easier to use and understand than a simultaneous multiple round auction (SMRA). Moreover, unlike in an SMRA, the auctioneer can specify how the quantities to be sold will depend on the auction prices, by choosing supply functions across varieties. Related material is at www.paulklemperer.org.

The Bank of England's implementations

I invented the auction for the Bank of England when it consulted me at the beginning of the financial crisis in 2007. It now uses it regularly to auction loans of funds secured against different varieties of collateral. The Bank runs the auction more often at times of potential stress. (It used the auction weekly after the 2016 "Brexit" vote.)

The Bank's first implementation corresponds closely to the description in section 2, below. The current implementation permits more dimensions (i.e., more "varieties" of goods), and also allocates an endogenous total quantity. We now use simple linear-programming methods to solve the auction.

The then-Governor, Mervyn King, told *the Economist* that "[the Product-Mix Auction] is a marvellous application of theoretical economics to a practical problem of vital importance to financial markets" (here).

The current Governor, Mark Carney, announced plans for greater use of the auction.

The *Guardian* newspaper published a 5 minute video including an interview with then-Deputy Governor, Paul Tucker here: https://www.theguardian.com/science/video/2013/jul/12/geometry-banking-crisis-video

The Corporate Bond Purchase Scheme that the Bank ran in 2016-17 also has some features in common with the auction described below.

The "Product-Mix Auction" is a single-round auction that can be used whenever an auctioneer wants to buy or sell multiple differentiated products. So potential applications include spectrum sales, and selling close-substitutes "types" of energy, as well as applications in finance such as the one discussed below.

The Product-Mix Auction effectively takes multiple single-round auctions that are each for (many units of) a single variety of the good, and combines all these auctions into one auction. Bidders express their relative preferences *between* varieties, as well as for alternative quantities of specific varieties, in simultaneous "sealed bids". All the information from all the bids is then used, in conjunction with the auctioneer's own preferences, to set *all* the prices *and* the total quantities of all the goods allocated.

The Product–Mix Auction is like the "simultaneous multiple-round auction" (SMRA) in that both auctions find competitive equilibrium allocations consistent with the preferences that participants express in their bids. However, the Product–Mix Auction has several advantages over the SMRA. It is (obviously) much faster than an SMRA's iterative process (for example, the UK's 3G mobile-phone SMRA auction took 150 rounds over seven weeks), and it is therefore also simpler to use and less vulnerable to collusion. Moreover, unlike a standard SMRA, it allows the auctioneer as well as the bidders, to specify how the relative quantities of the different varieties to be sold should depend on their relative prices.

I invented the auction at the beginning of the financial crisis, in response to the 2007 Northern Rock bank run--Britain's first bank run since the 1800's. The Bank of England has used it regularly and successfully since, to improve its allocation of funds to banks, building societies, etc. In the Bank's auction, the different "types of goods" are loans of funds secured against different types of collateral, and the "prices" are interest rates, that is, the interest rate a borrower pays depends on the quality of the collateral it offers.[1]

One indication of the auction's success is that the then-Governor of the Bank of England (Mervyn King) wrote, after the Bank had been using it regularly for over eighteen months and auctioned £80 billion worth of repos using it, that "the Bank of England's use of Klemperer auctions in our liquidity insurance operations is a marvellous application of theoretical economics to a practical problem of vital importance to financial markets"; he made a similar statement to the *Economist* a year later; and an Executive Director of the Bank described the auction as "A world first in central banking...potentially a major step forward in practical policies to support financial stability".[2]

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[1] High-quality collateral would include UK sovereign debt; poorer collateral would include mortgage-backed securities.

[2] A Deputy Governor's remarks were published by a major British newspaper (the *Guardian*) here. See Bank of

The Bank of England is enthusiastic about the way not only the prices (i.e., interest rates) but also-- by contrast with the SMRA-- the *quantities* of the different goods (loans against different collaterals) are determined within the auction, as a function both of bids on all the collaterals, and of its own preferences (expressed through a set of "supply functions"). Specifically, the Bank doesn't want to accept much low-quality collateral unless the markets are stressed, so the quantities of poorer collaterals that its auction accepts are increasing functions of the differences between the bids offering them and the bids offering better collaterals.

So the Product-Mix Auction is better than many central banks' approaches of treating all collateral as identical, or pre-determining the interest-rate spreads between different collaterals. Fixed price differences take no account of the auctioneer's preferences about the relative amounts of each variety it would like to sell. Nor do fixed price differences make any use of the market's information about the "correct" relative prices for different collaterals—a central bank often has very imperfect information, and even when it has some information, it might not want to reveal its views to the market. Finally, the auction gives the central bank useful insight into the market's views about relative prices.

The Product-Mix Auction is also superior to, e.g., the U.S. Federal Reserve's 2008-9 Term Securities Lending Facility (TSLF) that held separate auctions for loans against different types of collateral in different weeks. Running separate auctions means neither the quantity of funds allocated, nor the interest rate, on any collateral can depend on the bids on *other* collaterals. Moreover, by handling multiple goods simultaneously, the Product-Mix Auction gives bidders less market power and less ability to manipulate outcomes than if the goods were auctioned separately. So outcomes better approximate the perfectly competitive ideal.

The Product-Mix Auction approach also helps bidders. For example, a commercial bank might be interested in borrowing £500 million pounds, and be willing to pay up to 3% interest if it could use poor collateral, but prefer to use good collateral if that reduced its interest rate by 1% or more. Such a bidder can make a bid expressing these preferences—and the auction will then always do what is best for the bidder given the final auction prices. So, by contrast with the TSLF, such a bidder never has to guess which auction to bid in.

Crucially, the auction is simple to understand. The bidding rules, and the general principles of the auction, are easily comprehensible. Moreover, in the two-variety case the diagrammatic solution method is simple enough that the top officials of the Bank were quickly able to understand it, and so became comfortable that the approach was robust. The simplicity of the auction was critical both to

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England (2010, 2011), Fisher (2011), Milnes (2010), Fisher, Frost and Weeken (2011), Frost, Govier, and Horn (2015), the *Economist* (2012), and the Bank of England's website.

getting the auction accepted for practical use, and to it working efficiently (i.e., bidders finding it easy to participate, and also choosing their bids correctly).

The fact that the diagrammatic solution method creates and intersects (relative) demand and supply curves to find the competitive equilibrium, and that this then specifies the auction's solution, also makes it appealing to practitioners who have basic economics training. Indeed, the visualisations that the Bank of England uses during each auction emphasises the familiar economic logic of efficient competitive equilibrium. In the Bank's initial two-variety implementation the (relative) demand curve created from the bids was shown superimposed on the Bank's supply curve (as illustrated in figure 2 below). That visualisation therefore looked exactly like the standard supply-and-demand diagram that demonstrates competitive equilibrium in an elementary textbook (although we will see that the Bank could immediately read off *both* final quantities, as well as the final price difference between the two goods, off this diagram).

More than two varieties requires more visualisations. (For the second-generation implementation, the Bank set up four simple charts to guide them during the auction.) However, experience of implementing and using the simple two-variety case was crucial in making the Bank's officials comfortable about updating the auction to the many-variety case.[3]

Another recent modification to the Bank's auction means that the *total* quantity allocated (summed over all goods) now responds in a more sophisticated way to the information in the bidding. And although the auction was designed in response to the crisis, the Bank wanted a solution that would be used in normal times too (in part, so that the use of a specific auction design would convey no information). So the auction is now used regularly; indeed the current-Governor, Mark Carney, has announced plans for its greater use.[4]

The remainder of the main text is a tiny revision of my 2008 paper. After the introduction (section 1), section 2 describes a two-good implementation that corresponds closely to the Bank of

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[3] With more than two varieties a purely graphical solution method would be both inefficient and hard to understand, so the Bank of England uses a simple linear programme to solve the many-variety case, plus a set of visualisations to describe the solution. (See Frost, Govier, and Horn (2015) and—especially--Baldwin and Klemperer (in preparation, a,b) for more details.)

[4] The auctions were initially monthly for up to £5 billion of 3 month repos (or £2.5 billion for 6 month repos). The auctions are now monthly for up to a potentially much larger total quantity which itself depends in a sophisticated way on the bidding, of 6 month repos. (More frequent auctions have been held at times of potential stress; they were held weekly after the 2016 "Brexit" vote.) Actual quantities have varied enormously depending on market conditions and other Central Bank policies. As an illustration, in 2010-11, the spreads between high and low quality collateral varied between 9 and 30 bps for 3 month repos; between 16 and 53 bps for 6 month repos.

England's initial implementation.[5] Section 3 describes easy extensions.[6] Section 4 describes further extensions and the relationship to the SMRA, and section 5 concludes.

Appendix I describes different "languages" in which bidders can express their preferences in more detail. Appendix II gives a little more information about Iceland's planned implementation.

1. Introduction

How should goods that both seller(s) and buyers view as imperfect substitutes be sold, especially when multi-round auctions are impractical?

This was the Bank of England's problem in autumn 2007 as the credit crunch began.[7] The Bank urgently wanted to supply liquidity to banks, and was therefore willing to accept a wider-than-usual range of collateral, but it wanted a correspondingly higher interest rate against any weaker collateral it took. A similar problem was the U.S. Treasury's autumn 2008 Troubled Asset Recovery Program (TARP) plan to spend up to $700 billion buying "toxic assets" from among 25,000 closely-related but distinct sub-prime mortgage-backed securities.

Because financial markets move fast, in both cases it was highly desirable that any auction take place at a single instant. In a multi-stage auction, bidders who had entered the highest bids early on might change their minds about wanting to be winners before the auction closed,[8] and the financial markets might themselves be influenced by the evolution of the auction, which magnifies the difficulties of bidding and invites manipulation.[9]

An equivalent problem is that of a firm choosing its "product mix": it can supply multiple varieties of a product (at different costs), but with a total capacity constraint, to customers with different preferences between those product varieties, and where transaction costs or other time pressures make

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[5] I do *not* give full details of the Bank's objectives and constraints here, and not all the issues I discuss are relevant to it. See Bank of England (2010, 2011), Fisher (2011), Fisher, Frost and Weeken (2011), and the Bank of England's website.

[6] In 2014 we extended the Bank of England's auction to more goods (and used a different solution method that is simpler for the many-good case) and to variable total quantity (section 3.3); future auctions may use others of the enhancements described in section 3. See Frost, Govier, and Horn (2015), the Bank of England's website, and--especially--Baldwin and Klemperer (in preparation, a).

[7] The crisis began in early August 2007, and a bank run led to Northern Rock's collapse in mid-September. Immediately afterwards, the Bank of England first ran four very-unsuccessful auctions to supply additional liquidity to banks and then consulted me. I got valuable assistance from Jeremy Bulow and Daniel Marszalec.

[8] Some evidence is that most bids in standard Treasury auctions are made in the last few minutes, and a large fraction in the last few seconds. For a multi-round auction to have any merit, untopped bids cannot be withdrawn without incurring penalties.

[9] The Bank of England insisted on a single stage auction. Ausubel and Cramton (2008) argued a multi-stage auction was feasible for the U.S. Treasury.

multiple-round auctions infeasible.[10] The different varieties of a product could include different points of delivery, different warranties, or different restrictive covenants on use.

This paper outlines a solution to all these problems – the Product-Mix Auction, which I invented for the Bank of England, and also subsequently proposed to the U.S. Treasury.[11]

My design is straightforward in concept – each bidder can make one or more bids, and *each* bid contains a *set* of mutually exclusive offers. Each offer specifies a price (or, in the Bank of England's auction, an interest-rate) for a quantity of a specific "variety". The auctioneer looks at all the bids and then selects a price for each "variety". From each bid offered by each bidder, the auctioneer accepts (only) the offer that gives the bidder the greatest surplus at the selected prices, or no offer if all the offers would give the bidder negative surplus. All accepted offers for a variety pay the same (uniform) price for that variety.

The idea is that the menu of mutually-exclusive sets of offers allows each bidder to approximate a demand function, so bidders can, in effect, decide how much of each variety to buy *after* seeing the prices chosen. Meanwhile the auctioneer can, if it wishes, look at demand *before* choosing the prices; allowing it to choose the prices ex-post creates no problem here, because it allocates each bidder precisely what that bidder would have chosen for itself given those prices.[12] (In practice, the Bank of England precommits to a supply function that will uniquely determine the prices as a function of the bids.[13])

Importantly, offers for each variety provide a competitive discipline on the offers for the other varieties, because they are all being auctioned simultaneously.

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[10] That is, the Bank of England can be thought of as a "firm" whose "product" is loans; the different "varieties" of loans correspond to the different collaterals they are made against, and their total supply may be constrained. The Bank's "customers" are its counterparties, and the "prices" they bid are interest rates.

[11] After I had proposed my solution to the Bank of England, I learned that Paul Milgrom was independently pursuing related ideas. He and I therefore made a joint proposal to the U.S. Treasury, together with Jeremy Bulow and Jon Levin, in September-October 2008. Other consultants, too, proposed a static (sealed-bid) design, although of a simpler form, and the Treasury planned to run a first set of simple sealed-bid auctions, each for a related group of assets, and then enhance the design using some of the Bulow-Klemperer-Levin-Milgrom ideas in later auctions. However, it then suddenly abandoned its plans to buy subprime assets (in November 2008). Note also, however, that Larry Ausubel and Peter Cramton (who played an important role in demonstrating the value of using auctions for TARP, see e.g., Ausubel et al. (2008)) had proposed running dynamic auctions, and the possibility of doing this at a later stage was also still being explored.

Milgrom (2009) shows how to represent a wide range of bidders' preferences such that goods are substitutes, and shows a linear-programming approach yields integer allocations when demands and constraints are integer, but my proposal seems more straightforward and transparent in a context such as the Bank of England's.

[12] That is, it chooses prices like a Walrasian auctioneer who is equating bidders' demand with the bid-taker's supply in a decentralized process (in which the privately-held information needed to determine the allocation is directly revealed by the choices of those who hold it).

The result assumes the conditions for "truthful" bidding are satisfied – see below.

[13] In the many-variety version of the auction in use since 2014, the Bank of England precommits to a *set of* supply functions.

Compare this with the "standard" approach of running a separate auction for each different "variety". In this case, outcomes are erratic and inefficient, because the auctioneer has to choose how much of each variety to offer before learning bidders' preferences, and bidders have to guess how much to bid for in each auction without knowing what the price-differences between varieties will turn out to be; the wrong bidders may win, and those who do win may be inefficiently allocated across varieties. Furthermore, each individual auction is much more sensitive to market power, to manipulation, and to informational asymmetries, than if all offers compete directly with each other in a single auction. The auctioneer's revenues are correspondingly generally lower.[14] All these problems also reduce the auctions' value as a source of information. They may also reduce participation, which can create "second-round" feedback effects furthering magnifying the problems.[15]

Another common approach is to set fixed price supplements for "superior" varieties, and then auction all units as if they are otherwise homogenous. This can sometimes work well, but such an auction cannot take any account of the auctioneer's preferences about the proportions of different varieties transacted.[16] Furthermore, the auctioneer suffers from adverse selection.[17]

Section 2 shows how the Product-Mix Auction implements my alternative approach in a way that is simple, robust, and easy to understand, so that bidders are happy to participate. Section 3 discusses extensions. In particular, it is easy to include multiple buyers and multiple sellers, and "swappers" who may be on either, or both, sides of the market. Section 4 compares the Product-Mix Auction with the simultaneous multiple-round auction (SMRA). The Product-Mix Auction yields the same results as a "proxy" SMRA whose implementation is modified to permit the auctioneer to specify how the quantities traded will depend on the final prices. The Product-Mix Auction's static (i.e., single round of bids) design means it is also simpler, cheaper, and less susceptible to collusion and other abuses of market power, than is a (standard dynamic) SMRA. Section 5 concludes.

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[14] Thus, for example, if the U.S. Treasury had simply predetermined the amount of each type of security to purchase, ignoring the information about demand for the large number of closely-related securities, competition would have been inadequate. There were perhaps 300 likely sellers, but the largest 10 held of the order of two-thirds of the total volume, and ownership of many individual securities was far more highly concentrated.

[15] The feedback effects by which low participation reduces liquidity, which further reduces participation and liquidity, etc., are much more important when there are multiple agents on both sides of the market -- see Klemperer (2008).

[16] Moreover, as noted above, a Central Bank might not want to signal its view of appropriate price-differentials for different collaterals to the market in advance of the auction.

[17] If, for example, the U.S. Treasury had simply developed a "reference price" for each asset, the bidders would have sold it large quantities of the assets whose reference prices were set too high -- and mistakes would have been inevitable, since the government had so much less information than the sellers.

2. A Simple Two-Variety Example

The application this auction was originally designed for provides a simple illustration. A single seller, the Bank of England (henceforth "the Bank") auctioned just two "goods", namely a loan of funds secured against strong collateral, and a loan of funds secured against weak collateral. For simplicity I refer to the two goods as "strong" and "weak".[18] In this context, a per-unit price is an interest rate. The rules of the auction are as follows:

1. Each bidder can make any number of bids. *Each bid* specifies a *single* quantity and an offer of a per-unit price for *each* variety. The offers in each bid are mutually exclusive.
2. The auctioneer looks at all the bids and chooses a minimum "cut-off" price for each variety – I will describe later in this section how it uses the construction illustrated in Figures 1a, 1b, and 2 to determine these minimum prices uniquely, for any given set of bids, and given its own preferences.
3. The auctioneer accepts all offers that exceed the minimum price for the corresponding variety, *except* that it accepts at most one offer from each bid. If both price-offers in any bid exceed the minimum price for the corresponding variety, the auctioneer accepts the offer that maximizes the bidder's surplus, as measured by the offer's distance above the minimum price.[19]
4. All accepted offers pay the minimum price for the corresponding variety – that is, there is "uniform pricing" for each variety.[20]

Thus, for example, one bidder might make three separate bids: a bid for £375 million at {5.95% for (funds secured against) weak OR 5.7% for (funds secured against) strong}; a bid for an additional £500 million at {5.75% for weak OR 5.5% for strong}; and a bid for a further £300 million at {5.7% for weak OR 0% for strong}. Note that since offers at a price of zero are never selected, the last bid is equivalent to a traditional bid on only a single collateral.[21]

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[18] We assume (as did the Bank) that there is no adverse selection problem regarding collateral. For the case in which bidders have private information regarding the value of the collateral they offer, see Manelli and Vincent (1995).

[19] See notes 22 and 25 for how to break ties, and ration offers that equal the minimum price.

[20] Klemperer (2008) discusses alternative rules. In particular, the Product-Mix Auction could be run with discriminatory ("pay your bid") pricing. However, discriminatory pricing leads to less efficient outcomes than uniform pricing, even if bidders bid optimally. Furthermore, because it is harder for new bidders to understand how to bid, discriminatory pricing may discourage entry of needy banks in times of crisis. Moreover, the information that the auction provides the central bank is better with uniform pricing. The reason is that with uniform pricing, if the number of bidders is not too small, a bidder that knows its own valuation should choose bids that (approximately) represent its true preferences (see the last paragraph of this section, and note 27). The collateral a bidder is allocated on any bid is then always the one that is best for it given the prices the auction sets. With discriminatory pricing, optimal bidding is not close to "truthful", and would generally not always result in the bidder being allocated the collateral that is best for it if both price-offers in a bid exceed the minimum price for the corresponding variety. So the Product-Mix Auction's "either/or" bids exacerbate the difficulties that discriminatory pricing can cause, relative to uniform pricing.

[21] A bidder can, of course, restrict each of its bids to a single variety. Note also that a bidder who wants to guarantee winning a fixed total quantity can do so by making a bid at an arbitrarily large price for its preferred variety, and at an appropriate discount from this price for the other variety.

An example of the universe of all the bids submitted by all the bidders is illustrated in Figure 1a. The prices (i.e., interest rates) for weak and strong are plotted vertically and horizontally, respectively; each dot in the chart represents an "either/or" bid. The number by each dot is the quantity of the bid (in £millions). The three bids made by the bidder described above are the enlarged dots highlighted in bold.
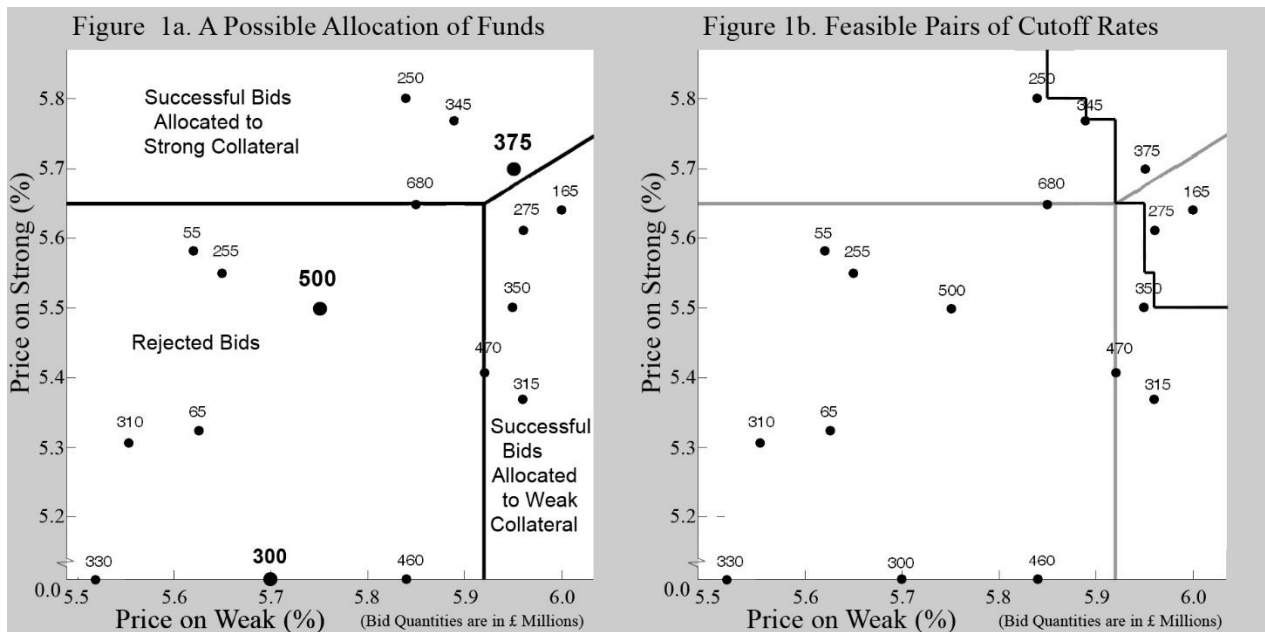

Figure 1: An example of bids in the Bank of England's auction.

The cut-off prices and the winning bids are determined by the Bank's objectives. If, for example, the Bank wants to lend £2.5 billion, and there are a total of £5.5 billion in bids, then it must choose £3 billion in bids to reject.

Any possible set of rejected bids must lie in a rectangle with a vertex at the origin. Figure 1a shows one possible rectangle of rejected bids, bounded by the vertical line at 5.92% and the horizontal line at 5.65%. If the Bank were to reject this rectangle of bids, then all the accepted bids -- those outside the rectangle -- would pay the cut-off prices given by the boundaries: 5.92% for weak, and 5.65% for strong.

Bids to the north-east of the rectangle (i.e, those which could be accepted for either variety) are allocated to the variety for which the price is further below the offer. So bids that are both north of the rectangle, and north-west of the diagonal 45° line drawn up from the upper-right corner of the rectangle, receive strong, and the other accepted bids receive weak.

Of course, there are many possible rectangles that contain the correct volume of bids to reject. On any 45° line on the plane, there is generally exactly one point that is the upper-right corner of such a rectangle.[22] It is easy to see that the set of all these points forms the stepped downward–sloping line shown in Figure 1b.[23] This stepped line is therefore the set of feasible pairs of cut-off prices that accept exactly the correct volume of bids.

Every point on Figure 1b's stepped line (i.e., every possible price pair) implies both a price-difference and (by summing the accepted bids below the corresponding 45° line) a proportion of sales that are weak. As the price-difference is increased, the proportion of weak sales decreases. Using this information we can construct the downward-sloping "demand curve" in Figure 2.
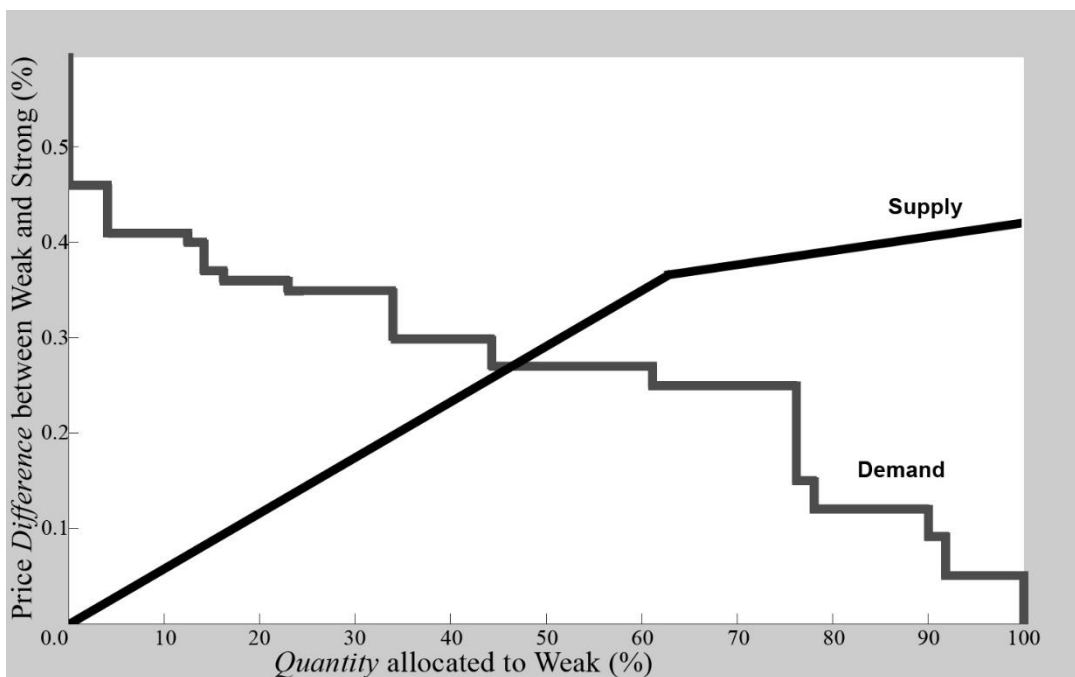

Figure 2: Equilibrium in the Bank of England's auction.

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[22] Moving north-east along any 45° line represents increasing all prices while maintaining a constant difference between them. Because the marginal bid(s) is usually rationed, there is usually a single critical point that rejects the correct volume of bids. But if exactly £3 billion of bids can be rejected by rejecting entire bids, there will be an interval of points between the last rejected and the first accepted bid. As a tie-breaking rule, I choose the most south-westerly of these points.

[23] The initial vertical segment starts at the highest price for weak such that enough can be accepted on weak when none is accepted on strong (this price is the weak price of the bid for 680), and continues down as far as the highest price bid for strong (the strong price of the bid for 250). At this point some strong replaces some weak in the accepted set, and there is then a horizontal segment until we reach the next price bid for weak (the weak price of the bid for 345) where more strong replaces weak in the accepted set and another vertical segment begins, etc.

If it wished, the auctioneer (the Bank) could give itself discretion to choose any point on the "demand curve" (equivalently, any feasible rectangle in Figures 1a, 1b) after seeing the bids. In fact, the Bank prefers to precommit to a rule that will determine its choice. That is, the Bank chooses a "supply curve" or "supply schedule" such as the upward-sloping line in Figure 2 so the proportion allocated to weak increases with the price-difference.[24]

The point of intersection between the Bank's supply curve and the "demand curve" constructed from the bids determines the price differential and the percentage of weak sold in the auction. With the supply curve illustrated, the price difference is 0.27% and the proportion of weak is 45% -- corresponding to the outcome shown in Figure 1a.[25]

This procedure ensures that bidders whose bids reflect their true preferences[26] receive precisely the quantities that they would have chosen for themselves if they had known the auction prices in advance. So unless a bidder thinks its own bids will affect the auction prices, its best strategy is to bid "truthfully"; if bidders all do this, and the Bank's supply curve also reflects its true preferences, the auction outcome is the competitive equilibrium.[27]

3. Easy Extensions

3.1 Multiple buyers and multiple sellers

It is easy to include additional potential sellers (i.e., additional lenders of funds, in our example). Simply add their maximum supply to the total that the auctioneer sells, but allow them to participate in the

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[24] The proposal for the U.S. TARP to employ a "reference price" for each asset corresponds to choosing the multi-dimensional equivalent of a horizontal supply curve; buying a predetermined quantity of each asset corresponds to using a vertical supply curve. As I noted above, both these approaches are flawed. Choosing an upward-sloping supply curve maintains the advantage of the reference-price approach, while limiting the costs of mispricing. (The optimal choice of supply-curve slope involves issues akin to those discussed in Poole (1970), Weitzman (1974), Klemperer and Meyer (1986), etc.; maintaining the reserve power to alter the supply curve after seeing the bids protects against collusion, etc., see Klemperer and Meyer (1989), Kremer and Nyborg (2004), Back and Zender (2001), McAdams (2007), etc.)

[25] By determining the proportion of weak, Figure 2 also determines what fractions of any bids on the rectangle's borders are filled, and the allocation between goods of any bids on the 45° line.

[26] This does not require pure "private value" preferences, but does not allow bidders to change their bids in response to observing others' bids.

We can extend our mechanism to allow bidders with "common values" to update their bids: the auctioneer takes bids as above, and reports the "interim" auction prices that would result if its supply were scaled up by some pre-determined multiple (e.g., 1.25). It then allows bidders to revise the prices of any bid that would win at the interim prices, except that the price on the variety that the bid would win cannot be reduced below that variety's interim price. Multiple such stages can be used, and/or more information can be reported at each stage, before final prices and allocations are determined -- we offered such an option to the U.S. Treasury, though it was not our main recommendation.

[27] Because on the order of 40 commercial banks, building societies, etc., bid in the Bank of England's auctions, it is unlikely that any one of them can much affect the prices. I assume the Bank's supply curve is upward sloping so, given our tie-breaking rule (see note 22), if there are multiple competitive equilibria the outcome is the unique one that is lowest in both prices.

auction as usual. If a potential seller wins nothing in the auction, the auctioneer has sold the seller's supply for it. If a potential seller wins its total supply back, there is no change in its position.

3.2 "Swappers" who might want to be on either side of the market

Exactly the same approach permits a trader to be on either, or both, sides of the market. If, for example, letting the auctioneer offer its current holdings of strong, a bidder in the auction wins the same amount of weak, it has simply swapped goods (paying the difference between the market-clearing prices).

3.3 Variable total quantity

Making the total quantity sold (as well as the proportions allocated to the different varieties) depend upon the prices is easy. (The Bank of England's 2014 implementation does this.)

The auctioneer might, for example, precommit to the total quantity being a particular increasing function of the price of strong. Using the procedure of Section 2 to solve for the strong price corresponding to every possible total quantity yields a weakly-decreasing function, and the unique intersection of the two functions then determines the equilibrium.

3.4 Other easy extensions

There can, of course, be more than two goods, with a cut-off price for each, and a bid rejected only if *all* its offers are below the corresponding cut-off prices.

Bidders can also be allowed to ask for different amounts of the different goods in a bid.

Relatedly, Iceland planned a Product-Mix Auction in which each of a bidder's bids could specify cut-off prices for three alternative goods, and the amount of money to be spent by the bid. So the amount of a good bought by a winning bid would be the amount to be spent divided by the chosen good's price; the type of good (if any) bought would depend on all goods' prices, as usual. (This version of the auction may be useful when, e.g., a country's creditors exchange their claims for a choice of one or more new debt instruments, or an acquired firm's shareholders exchange their shares for a choice of cash or shares in the acquiring firm. See Appendix II for details and discussion.)

Bidders can be permitted to specify that a total quantity constraint applies across a group of bids.

Several other extensions are also easy.

Importantly, bidders can express more complex preferences by using several bids in combination: for example, a bidder might be interested in £100 million weak at up to 7%, and £80 million strong at up to 5%. However, even if prices are high, the bidder wants an absolute minimum of

£40 million. This can be implemented by making all of the following four bids, if negative bids are permitted:

1. £40 million of {weak at maximum permitted bid OR strong at maximum permitted bid *less* 2%}.
2. £100 million of weak at 7%.
3. £80 million of strong at 5%.
4. *minus* £40 million of {weak at 7% OR strong at 5%}.

The point is that the fourth (negative) bid kicks in exactly when one of the second and third bids is accepted, and then exactly cancels the first bid for £40 million "at any price" (since 2% = 7% - 5%).[28] Figure 5d in Appendix I illustrates these four bids, and how they together express the required preferences.

Appendix I discusses the different "languages" that bidders can use to express preferences of the different forms described in this subsection. This Appendix also illustrates how these languages can be represented geometrically – although bidders submit bids as lists of vectors, the ability to present these bids (and also the aggregate of all bidders' bids) geometrically helps both participants and analysts understand the auction.[29]

4. Further Extensions, and the Relationship to the Simultaneous Multiple Round Auction

My auction is equivalent to a static (sealed-bid) implementation of a simplified version of a "two-sided" simultaneous multiple round auction (SMRA). (By "two-sided" I mean that sellers as well as buyers can make offers – see below.)

Begin by considering the special case in which the auctioneer has predetermined the quantity of each variety it wishes to offer, and the bids in my auction represent bidders' true preferences. Then the outcome will be exactly the same as the limit as bid increments tend to zero of a standard SMRA if each

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[28] A bidder can perfectly represent any preferences across all allocations by using an appropriate pattern of positive and negative bids if the goods are imperfect substitutes such that the bidder's marginal value of a good is reduced at least as much by getting an additional unit of that good as by getting an additional unit of the other good (i.e., if $V(w,s)$ is the bidder's total value of £w of weak plus £s of strong, then $\partial^2 V / \partial w^2 \leq \partial^2 V / \partial w \partial s \leq 0$ and $\partial^2 V / \partial s^2 \leq \partial^2 V / \partial w \partial s \leq 0$). More general preferences than this require more complex representations--but the important point, of course, is that preferences can typically be well-approximated by simple sets of bids.
[29] The geometric techniques used in the analysis of Product-Mix Auctions also yield new results in the multidimensional analysis of demand: see Baldwin and Klemperer (2012, forthcoming).

bidder bids at every step to maximize its profits at the current prices given those preferences,[30] since both mechanisms simply select the competitive-equilibrium price vector.[31]

The general case in which the auctioneer offers a general supply curve relating the proportions of the different varieties sold to the price differences is not much harder. We now think of the auctioneer as acting *both* as the bid-taker selling the maximum possible quantity of both varieties, *and* as an additional buyer bidding to buy units back to achieve a point on its supply curve. That is, in our example in which the Bank auctions £2.5 billion, we consider an SMRA which supplies £2.5 billion weak *and* £2.5 billion strong, and we think of the Bank as an additional bidder who has an inelastic total demand for £2.5 billion and who bids in exactly the same way as any other bidder.[32] [33]

So my procedure is equivalent to a "proxy SMRA", that is, a procedure in which bidders submit their preferences, and the auctioneer (and other potential sellers) submit their supply curves, and a computer then calculates the equilibrium that the (two-sided) SMRA would yield.[34] However, my procedure restricts the preferences that the auction participants can express. Although I can permit more general forms of bidding than those discussed above (see Klemperer (2008)),[35] some constraints are desirable. For example, I am cautious about allowing bids that express preferences under which varieties are complements. (The Bank of England's 2014 implementation permits the auctioneer to express some complementary preferences.)[36]

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[30] In a SMRA the bidders take turns to make bids in many ascending auctions that are run simultaneously (e.g, 55% of 2.5 billion = 1.375 billion auctions for a single £1 of strong, and 45% of 2.5 billion = 1.125 billion auctions for a single £1 of weak). When it is a bidder's turn, it can make any new bids it wishes that beat any existing winning bid by at least the bidding increment (it cannot top up or withdraw any of its own existing bids). This continues until no one wants to submit any new bids. (For more detail, including "activity rules" etc., see, e.g., Milgrom (2000), Binmore and Klemperer (2002), and Klemperer (2004).)

[31] An exception is that an SMRA may not do this when bidders' preferences are such that they would ask for different amounts of the different goods in a single bid in my procedure. All the other types of bids discussed above reflect preferences such that all individual units of all goods are substitutes for all bidders (so bidding as described above in an SMRA is rational behaviour if the number of bidders is large). I assume the auctioneer also has such preferences (i.e., the Bank's supply curve is upward sloping), so if there are multiple competitive equilibria, there is a unique one in which all prices are lowest and both mechanisms select it (see note 27 and Crawford and Knoer (1981), Kelso and Crawford (1982), Gul and Stacchetti (1999), and Milgrom (2000)).

[32] That is, whenever it is the Bank's turn to bid, it makes the minimum bids to both restore its quantity of winning bids to £2.5 billion and win the quantity of each variety that puts it back on its supply curve, given the current price-difference. (It can always do this to within one bid increment, since the weak-minus-strong price difference can only be more (less) than when it last bid if its weak (strong) bids have all been topped, so it can increase the quantity of strong (weak) it repurchases relative to its previous bids, as it will wish to do in this case.)

[33] If there are other sellers (or "swappers") add their potential sales (or "swaps") to those offered in the SMRA, and think of these participants as bidding for positive amounts like any other bidders.

[34] Although Section 2's description may have obscured this, our procedure is symmetric between buyers and sellers. (It is not quite symmetric if the auctioneer doesn't precommit to its supply curve, but if bidders behave competitively their bids are unaffected by this.)

[35] I could in principle allow any preferences subject to computational issues; these issues are not very challenging in the Bank of England's problem.

[36] The difficulty with complements is the standard one that there might be multiple unrankable competitive equilibria, or competitive equilibrium might not exist (see note 31), and an SMRA can yield different outcomes depending upon the order in which bidders take turns to bid. In independent work, Milgrom (2009) explores how to restrict bidders to expressing "substitutes

Importantly, exercising market power is much harder in my procedure than in a standard SMRA, precisely because my procedure does not allow bidders to express preferences that depend on others' bids. In particular, coordinated demand reduction (whether or not supported by explicit collusion) and predatory behaviour may be almost impossible. In a standard dynamic SMRA, by contrast, bidders can learn from the bidding when such strategies are likely to be profitable, and how they can be implemented – in an SMRA, bidders can make bids that signal threats and offers to other bidders, and can easily punish those who fail to cooperate with them.[37][38]

Finally, the parallel with standard sealed-bid auctions makes my mechanism more familiar and natural than the SMRA to counterparties. In contexts like the Bank of England's, my procedure is much simpler to understand.

5. Conclusion

The Product-Mix Auction is a simple-to-use, sealed-bid, auction that allows bidders to bid on multiple differentiated assets simultaneously, and bid-takers to choose supply functions across assets. It can be used in environments in which a simultaneous multiple-round auction (SMRA) is infeasible because of transaction costs, or the time required to run it. The design also seems more familiar and natural than the SMRA to bidders in many applications, and makes it harder for bidders to collude or exercise market power in other ways.

Relative to running separate auctions for separate goods, the Product-Mix Auction yields better "matching" between suppliers and demanders, reduced market power, greater volume and liquidity, and therefore also improved efficiency, revenue, and quality of information. Its applications therefore extend well beyond the financial contexts for which I developed it.

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preferences". Crawford (2008)'s static mechanism for entry-level labor markets (e.g., the matching of new doctors to residency positions at hospitals) adddresses related issues in a more restrictive environment. See also Budish (2004).

[37] In a standard SMRA, a bidder can follow "collusive" strategies such as "I will bid for (only) half the lots if my competitor does also, but I will bid for more lots if my competitor does", see, e.g., Klemperer (2002, 2004), but in our procedure the bidder has no way to respond to others' bids. Of course, a bidder who represents a significant fraction of total demand will bid less than its true demand in *any* procedure, including mine, which charges it constant per-unit prices. But it is much easier for a bidder to (ab)use its market power in this way in an SRMA.

[38] A multi-round procedure (either an SMRA, or an extension of our procedure – see note 26) may be desirable if bidders' valuations have important "common-value" components, but may discourage entry of bidders who feel less able than their rivals to use the information learned between rounds.

Appendix I (*added 2018*): Bidding Languages for Product-Mix Auctions

This Appendix, illustrating and updating the original paper (Klemperer, 2008), was written in 2018 by Elizabeth Baldwin and Paul Klemperer.
Figs. 3-5 are based on those shown by Klemperer to Bank of England staff in 2008; Fig. 6 is based on one drawn by Klemperer for Iceland in 2015 (see Appendix II).

Different preference structures can be expressed using different variants of our "dot-bid" language—the choice between them should depend upon the context.[1] We describe several variants below, and illustrate them graphically for two-variety examples.[2]

In every variant, each *bidder* can make as many *bids* as she wishes.[3]

For concreteness, we call the good that is auctioned "fruit", and call its two varieties "apples" (A) and "bananas" (B). We write the per-unit prices that the auction sets as p_A and p_B (the same for every bid of every bidder), the quantities that the auction allocates to a given *bid* as q_A and q_B, and the total quantities that the auction allocates as Q_A and Q_B.[4] It will be convenient to write a *bidder*'s valuation of an apple as V(A), of two apples as V(2A), of an apple plus a banana as V(A+B), etc.

A suite of (free) programs to solve Product-Mix Auctions is available at http://pma.nuff.ox.ac.uk. The programs allow the choice among several alternative bidding languages (all those below except A2b); they also include variants that maximise the auctioneer's surplus (rather than efficiency as conventionally measured as the sum of the auctioneer's and the bidders' surpluses); and they also permit variants that use pay-as bid pricing or lowest-winner uniform pricing (rather than highest-loser uniform pricing), etc.

A. Tropical Languages

Languages that express quasi-linear preferences can be analysed using tropical-geometric techniques, see Baldwin and Klemperer (2012, forthcoming). So we call these languages "tropical".

A1a) the Bank of England Language

The original bidding language developed for, and implemented by, the Bank of England in response to the financial crisis, permitted simple "OR bids". Thus each bid is an offer that specifies a maximum per-unit price that the bidder is willing to pay for each variety, and a maximum quantity that the bidder is prepared to buy.

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[1] Participants must, of course, be able to express their preferences reasonably accurately, but the language must be easily comprehensible, so potential bidders (and the bid-taker!) feel comfortable participating, and there must be an algorithm to solve the auction, given the expressed preferences.
[2] We expect bidders to enter their bids as lists of vectors, but the ability to present both an individual bidder's bids, and the aggregate set of bids, graphically is important for helping participants understand and analyse the auction and the language.
[3] The auctioneer can, of course, restrict the number of bids permitted. The Bank of England's ILTR auction makes no restriction. At the other extreme, a student union's recent Product-Mix Auction of second-hand exercise machines restricted bidders to a single bid each (for reasons of equity). (In this special case, and with standard Product-Mix Auction pricing, i.e., "highest loser uniform pricing", the Product-Mix Auction is a Vickrey-Clark-Groves auction; this was the case for the union's auction even though they described it as a Product-Mix Auction.)
[4] Recall that the auction allocates each *bid* the optimal allocation, given the prices, for a bidder for whom this single bid accurately expresses her preferences.

For example, the bid (60, 50; 3) specifies that the bidder is willing to pay up to 60 cents for each apple, up to 50 cents for each banana, and is interested in buying up to 3 pieces of fruit. It can be represented graphically by the "dot-bid" shown as the solid black circle in fig. 3a, below; the desired quantity is shown in parentheses beside the dot-bid. The figure also shows the allocation desired by the bidder in the three different regions of price space. That is, this bid expresses the preferences that if $p_A<60$ and $60-p_A>50-p_B$, then the desired allocation is $q_A=3$, $q_B=0$; if $p_B<50$ and $50-p_B>60-p_A$, then the desired allocation is $q_A=0$, $q_B=3$; and if $p_A>60$ and $p_B>50$, then the desired allocation is $q_A=0$, $q_B=0$ (shown as $\emptyset$). (At any price vector on any line segment dividing the regions, the bidder is assumed to be indifferent among all convex combinations of the allocations desired in the adjacent regions.) In words, she wants to get "best value" from buying up to 3 items, given that she values each additional apple at 60 cents, and each additional banana at 50 cents. More concisely, the desired allocation is argmax_$\{q_A+q_B\leq3,\ q_A\geq0,\ q_B\geq0\}(q_A[60-p_A] + q_B[50-p_B])$.

Likewise fig. 3b illustrates the dot-bid (90, 25; 1) and the preferences expressed by that bid.
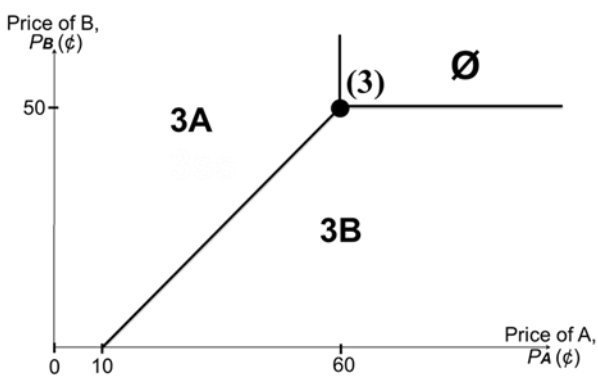

Fig. 3a: the bid (60, 50; 3)
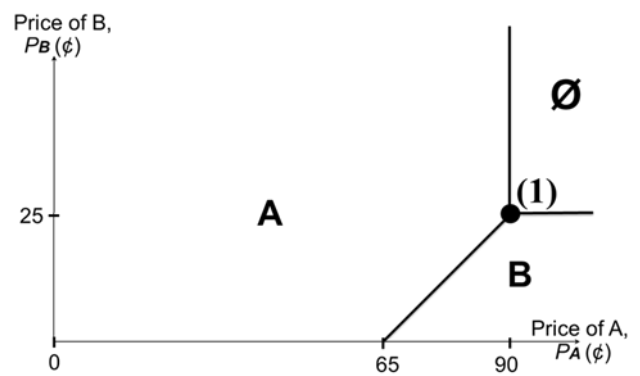and the preferences it expresses

Fig. 3b: the bid (90, 25; 1)
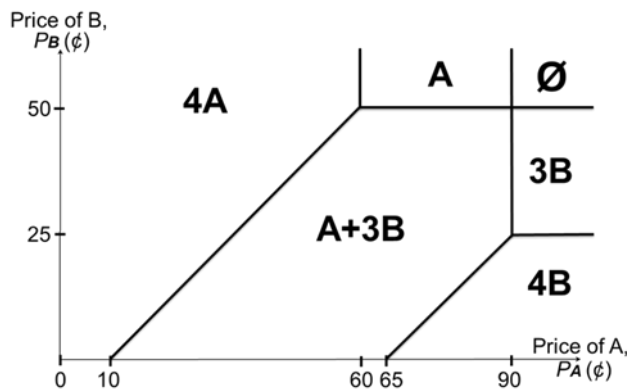and the preferences it expresses

Fig. 3c: the preferences expressed by the combination of the bids of figs. 3a and 3b

Fig. 3: Illustrations of the (ordinary) Bank of England language
(Regions of price space are labelled according
to the preferences expressed by the bids)

The preferences expressed by a bidder who makes *both* of the bids (60, 50; 3) and (90, 25; 1) can be found simply by superimposing figs. 3a and 3b. These preferences are shown in fig. 3c. (So it is obvious that from the auctioneer's point of view it is irrelevant whether a set of bids is made by a single bidder or by multiple

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bidders.[5]) Fig. 1a (of the main text) shows another example of dot-bids that might be made by either a single bidder, or a group of bidders.

A bidder can express more complex preferences by combining multiple bids. For example, the demand for "best value from up to 5 apples *and* up to 5 bananas *but* not more than 8 pieces of fruit in all" is achieved by making three separate bids: a bid for 3 apples (only), a bid for 3 bananas (only), and a bid for 2 pieces of fruit which can be either apples or bananas. (Thus, e.g., if the bidder is willing to pay up to 60 cents for each apple and up to 50 cents for each banana, the three required bids are (60, 0; 3), (0, 50; 3), and (60, 50; 2).)

Bidders can of course, if they wish, simply express a standard demand function for a single variety by submitting a list of bids at different prices; or they can do the same thing for multiple varieties. (In these cases, all the bids will be on the coordinate axes of the figure.)

The language generalises naturally to more than two varieties. For example, the bid (90, 25, 40; 2) specifies that the bidder is willing to pay up to 90 cents per apple, up to 25 cents per banana, up to 40 cents per clementine, and is interested in buying up to 2 pieces of fruit.[6]

The Bank of England language only allows the expression of "1-1 trade-offs". That is, any change in an agent's preferred allocation that results from an arbitrarily small change in prices must, generically, be either a change in her demand for just one variety or, if she switches between varieties, a substitution of a quantity of one variety for the same quantity of another (single) variety.[7] Moreover, not all "1-1 trade-offs" can be expressed—see section A2a, below. However, the restrictions that the Bank of England language imposes seem natural in many financial contexts.[8]

In the Bank of England's "ILTR" auctions, both the Bank and the bidders have been happy that this language gives the bidders more than enough freedom to express their preferences.[9] So although the Bank itself, acting as the auctioneer, now specifies its supply in a format that allows the expression of more complex preferences (see section E, below), it has not thus far extended any such facility to the bidders.[10]

A1b) Generalised Bank of England Language

Although the 1-1 trade-offs that the Bank of England language enforces are often natural, they might seem restrictive in some contexts. If, for example, the two varieties of fruit were apples and blueberries, a

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[5] This feature is important both for facilitating auction participants' understanding and also in academic analysis of the auction.

[6] The 2 pieces of fruit may be of the same variety. To bid for up to 2 pieces of fruit that must be of *different* varieties, see sec. C, below.

[7] A non-generic example is V(A)=V(B)=1; V(A+B)=2. (The bids corresponding to these preferences are (1,0; 1) and (0,1 ;1), and the graph representing these preferences (in the style of fig. 3) consists of a vertical and a horizontal line crossing at (1,1); increasing both prices from just below (1,1) to just above (1,1) reduces demand on both varieties from 1 unit to zero.)

[8] For example, a small fall in the interest rate for using weak collateral might mean a bidder substitutes a loan of £100 million using weak collateral for a loan of £100 million using strong collateral, but is unlikely to mean the bidder substitutes a loan of £200 million using weak collateral instead of a loan of £100 million using strong collateral.

   The Bank of England language permits the possibility that a further fall in interest rates using weak collateral might mean that the bidder wishes to take an additional loan of £100 million using weak collateral. So the bidder may, depending on the interest rates, choose a loan of £200 million using weak, or a loan of £100 million using weak, or a loan of £100 million using strong, or no loan. But the "1-1 trade-offs" assumption rules out the possibility that the *only* options that the bidder considers are a loan of £200 million using weak collateral, or a loan of £100 million using strong collateral, or no loan. (It also, of course, rules out any complements preferences.)

[9] Although many bidders each bid for multiple varieties in the same auction, it has almost always been by making a set of separate bids on more than one variety.

[10] The (free) program available at http://pma.nuff.ox.ac.uk for running auctions that use the Bank of England language solves a simple linear program to find the allocation of the good that maximises (conventionally measured) social welfare (since a competitive equilibrium does this).

consumer might respond to an increase in the price of apples by substituting many berries for 1 apple and, moreover, the number of berries she substitutes for the apple might be different at different prices.[11]

We can extend our language to deal with this case by defining a *generalised dot-bid* as a price vector together with an associated quantity *vector*. For example, fig. 4a shows the preferences expressed by a bidder who makes the bid (60, 40; 4, 5). Note that the gradient of the line that separates the region of price space in which the bidder's desired allocation is 4A from the region in which her desired allocation is 5B is the ratio of the elements of the quantity vector, namely 4/5. This reflects the fact that the bidder obtains the same utility from these two allocations, hence is indifferent between them, iff $4(60-p_A)= 5(40-p_B)$. (The equation of the indifference line is $p_B = (4/5)p_A - 8$.) Fig. 4b shows the preferences expressed by a bidder who makes both of the bids (60, 40; 4, 5) and (90, 25; 1, 3).
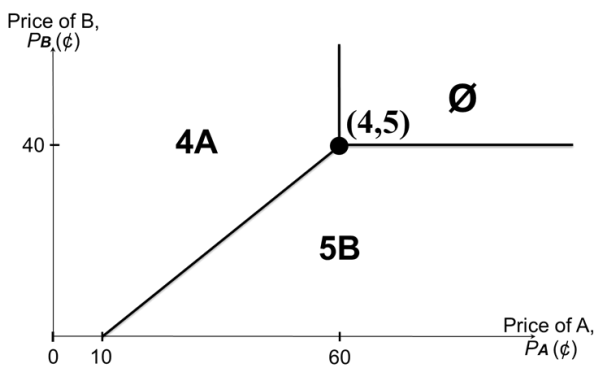

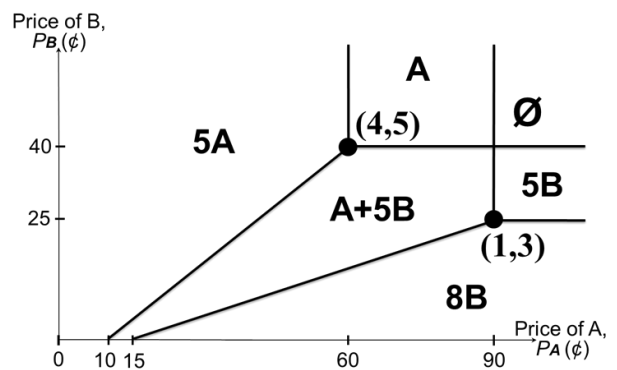Fig. 4a: the bid (60, 50; 4, 5) and the preferences it expresses

Fig. 4b: the bid of fig. 4a, together with the bid (90, 25; 1, 3), and the preferences expressed by this combination of bids

Fig. 4: Examples of the Generalised Bank of England Language

A2a) Strong-Substitutes Language

The Bank of England language, or its generalisation, seems sufficient for communicating bidders' preferences in many contexts, but an even more expressive language might sometimes be desirable.

In particular, if a bidder has marginal valuations for winning additional objects that depend upon which object(s) she has already won (and her valuations are not additively separable), her preferences cannot, in general (even if they involve only 1-1 trade-offs), be accurately expressed using ordinary dot-bids of the kind the Bank of England language permits.

Consider, for example, the preferences illustrated in fig. 5a. Here V(A)=70 and V(B)=55; V(2A)=95, V(2B)=75 and V(A+B)=100. So V(2A)-V(A) < V(A+B)-V(B) and V(2B)-V(B) < V(A+B)-V(A), that is, the bidder's marginal value of an apple is less if she already has an apple than if she already has a banana, and vice versa. Preferences such as these cannot be accurately expressed by ordinary dot-bids. (It is not hard to check this.) However, they are perfectly expressed by the combination of three ordinary (positive) dot-bids and one *negative (or "cancellation") dot-bid*, as illustrated in fig. 5b. The negative dot-bid is shown as a white circle, but for clarity we also indicate the sign of the dot-bid by signing the quantity bid for.

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[11] If the number of berries substituted for an apple is always the same, we can simply define the variety B as the number of berries that creates 1-1 trade-offs between A and B.

Negative dot-bids are treated by the auctioneer in exactly the same way as ordinary (positive) dot-bids. That is, a negative dot-bid is accepted whenever at least one of its prices exceeds the corresponding auction price and, if it is accepted, then it is allocated the variety on which its price exceeds the corresponding auction price by most. Since these dot-bids are for negative quantities, any accepted negative dot-bid simply cancels part or all of the demand created by positive dot-bids. (Note that negative dot-bids *cannot* be understood as offers to sell.[12])[13]
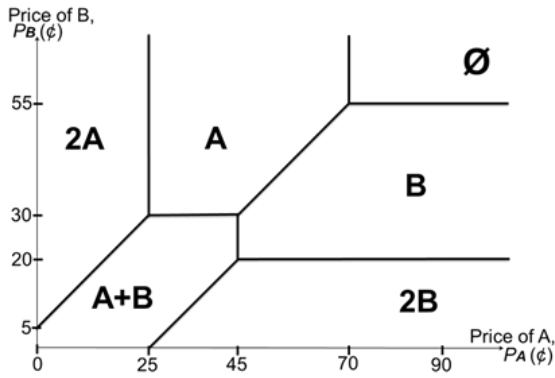

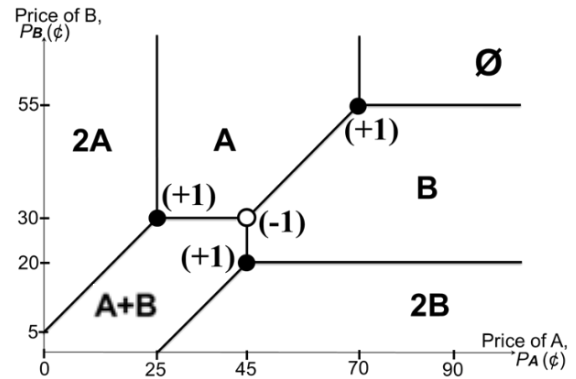Fig. 5a: Example of preferences that *cannot* be expressed in the Bank of England Language

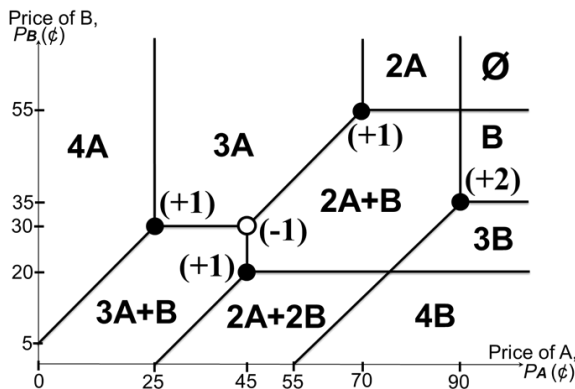Fig. 5b: Bids that express the preferences of fig.5a in the Strong Substitutes Language

Fig. 5c: The bids of fig.5a, together with the (tropical) bid (90, 35; 2), and the preferences expressed by this combination of bids
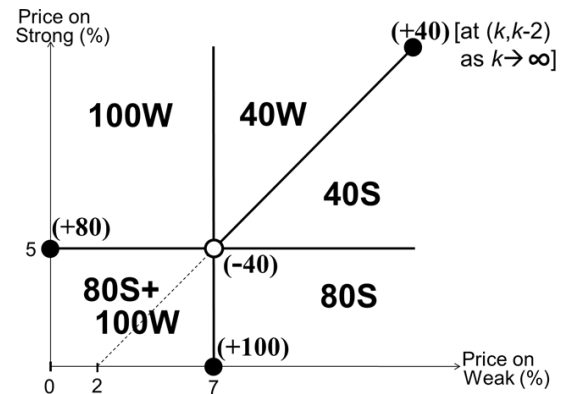
Fig. 5d: Bids that express the preferences of the example of sec. 3.4 of the main text in the Strong Substitutes Language

Fig. 5: Examples of the Strong Substitutes Language
(white circles represent negative (or "cancellation") bids)

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[12] A negative dot-bid cancels a purchase whenever one of its prices is sufficiently *high*, whereas an offer to sell would be accepted whenever one of its prices is sufficiently *low*.

[13] It is easy to construct fig. 5b (and similar figures) from the bidder's marginal values of additional units: the highest positive dot-bid (at the bottom left of the region where nothing is demanded) is at $(V(A)-0, V(B)-0) = (70, 55)$; the positive dot-bid at the bottom left of the region where A is demanded is at $(V(2A)-V(A), V(A+B)-V(A)) = (25, 30)$; the positive dot-bid at the bottom left of the region where B is demanded is at $(V(A+B)-V(B), V(2B)-V(B)) = (45, 20)$; and the negative dot-bid at the top right of the region where A+B is demanded is at $(V(A+B)-V(A), V(A+B)-V(B)) = (45, 30)$.

As another illustration if, when the bidder has already won $aA+bB$, for $(a,b) = (0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(0,2)$ and $(1,1)$, the marginal values of a single extra unit of (A, B) are (16,32), (4,24), (8,12), (2,23), (6,8) and (3,10), respectively, then there is a positive dot-bid (for +1 unit) at each of the 6 prices (16,32), (4,24), (8,12), (2,23), (6,8) and (3,10), and there is a negative dot-bid (for -1 unit) at each of the 3 prices (8,24), (3,23) and (6,10).

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Thus, for example, if the auction prices are p_A=30 and p_B=25, then the positive dot-bids (25, 30; 1), (45, 20; 1), and (70, 55; 1), are allocated 1 unit of B, 1 unit of A, and 1 unit of A, respectively. The negative dot-bid (45, 30; -1) is allocated *minus* 1 unit of A. So the total allocation to the entire set of bids is 1 unit of A plus 1 unit of B. If instead the auction prices are p_A=10 and p_B=25, the three positive dot-bids are all allocated 1 unit of A, and the negative dot-bid is again allocated *minus* 1 unit of A, so the total allocation to the set of bids is 2 units of A. And if the auction prices are p_A=50 and p_B=25, the positive dot-bid (45, 20; 1) is not allocated to either variety, the other two positive dot-bid are each allocated 1 unit of B, and the negative dot-bid is allocated *minus* 1 unit of B, so the total allocation to the set of bids is 1 unit of B. Etc.

As before, any quantities (not just +/- 1) can be associated with both positive and negative dot-bids; fig. 5c adds an additional dot-bid (90, 35; 2) to those in fig. 5b, and shows the preferences that are expressed by the aggregate of all these bids.

Using positive and negative dot-bids permits the expression of natural forms of preferences such as the example discussed in sec. 3.4 of the main text. In that example, a bidder is interested in a loan from the Bank of England of up to £80 million using strong collateral, and also up to £100 million using weak collateral, at interest rates up to 5% and 7%, respectively. However, it wants to be certain to receive a loan of at least £40 million (it is financially-distressed) so is willing to pay almost any price, using either collateral (but would pay 2% more to use weak than use strong), to ensure this. The set of dot-bids that precisely expresses these preferences are the three positive dot-bids (7, 0; 100m), (0, 5; 80m) and (k, k-2; 40m) (in which k is chosen "at ∞", that is, large enough that this bid is always selected), *plus* the negative dot-bid (7, 5; -40m). These bids are shown in fig. 5d.[14]

As for the Bank of England languages, we can always solve for the prices the auction sets as if all the dot-bids were made by a single bidder. So it remains true that from the auctioneer's point of view it is irrelevant which bid is made by which bidder. However, with negative dot-bids, care is sometimes needed in determining which marginal dot-bids to accept.[15]

Note also, however, that whereas any set of ordinary (positive) dot-bids represents a valid set of preferences, not all sets of positive and negative dot-bids represent valid preferences. (By valid we mean that money that is not spent in the auction has non-negative value, so "preferences" that imply that raising the price of just one variety means the bidder buys strictly more of it are not valid.) The (free) program for running the auction at http://pma.nuff.ox.ac.uk[16] also tests whether any given set of dot-bids represents valid preferences.**[17]**

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[14] I (Klemperer) discussed pictures like this one (and also alternative ways to describe them, cf. section C's discussion of "words") with Bank of England staff in 2008, but they felt such bids were unnecessary. (The Bank has, of course, other facilities for counterparties in need.)

[15] For example, one can't accept all a bidder's marginal positive bids at the same time as using her marginal negative bids to cancel positive bids from another bidder (e.g., if two bidders' sets of bids are superimposed). If all bids are positive (or if all the negative bids are non-marginal), there is only ever an issue if one wants to ration "fairly"– since any allocation of the bids is then valid.

[16] For the "Strong-Substitutes Language", the program uses Baldwin, Goldberg, Klemperer and Lock's (in prep.) algorithm to find the solution. An alternative method involves minimising the difference in the objective functions of two linear programs as we vary the total resource allocated to positive bids (and correspondingly vary the quantity cancelled by negative bids); the first of these linear programs is the one that we would solve if the positive dot-bids were the only bids (this program is discussed in note 10); the second linear program is the one we would solve if the negative dot-bids were the only bids but they all had positive sign.

[17] *This feature of the program is in preparation.* The examples of note 13 illustrate the necessary conditions for a set of dot-bids to represent valid preferences. Observe, in the first example, not only that the co-ordinates of the negative dot-bid are determined by the co-ordinates of the positive dot-bids, but also that the co-ordinates of the positive dot-bids must satisfy the condition [V(A+B)-V(A)] + [V(A)- V(0)] = [V(A+B)- V(B)] + [V(B)- V(0)] (here 30+70=45+55). In the second example, the co-ordinates of the negative dot-bids, and the co-ordinates 24, 3, and 10, are all determined by the remaining co-ordinates.

Theorem: *Any bidder with strong-substitutes preferences can perfectly represent her preferences using positive and negative dot-bids.* (This is the result stated for two varieties in Klemperer (2010); the generalisation to $n>2$ varieties is proved in Baldwin and Klemperer, in prep., b.)[18]

In the two-variety case, a bidder has "strong-substitute preferences" if and only if her marginal value of a unit of one variety is reduced at least as much by getting an additional unit of that variety as by getting an additional unit of the other variety (see note 28 of the main text). In fig 5a, for example, the marginal value of an apple to the bidder is 25 if she already has an apple, but 45 if she has a banana; the marginal value of a banana to the bidder is 20 if she already has a banana, but 30 if she has an apple.

More generally, having "strong-substitutes preferences" corresponds to having *any* preferences that imply "1-1 trade-offs" generically everywhere between different varieties of *indivisible* goods.[19] However, a bidder with any preferences that imply "1-1 trade-offs" everywhere between *divisible* goods can represent those preferences arbitrarily closely by using an arbitrarily large number of bids.[20]

A2b) the All-Substitutes language

The all-substitutes language generalises the strong-substitutes language exactly analogously to how the generalised Bank of England language generalises the (original) Bank of England language.

Theorem: *Any bidder with (any) substitutes preferences can perfectly represent her preferences using generalised positive and negative dot-bids.*[21] (See Baldwin and Klemperer, in prep., b)[22]

B. An Arctic Language

Iceland's planned auction permitted the expression of budget constraints (see Appendix II), which generate preferences which cannot be described tropically (since they are not quasilinear). So we call Iceland's language "arctic".

In this language, a dot-bid $(y_A,y_B; x)$ is a bid to spend a given *quantity of money*, x, at prices up to (y_A,y_B), and to be allocated variety A if $y_A/p_A > y_B/p_B$ & $p_A<y_A$, and variety B if $y_B/p_B > y_A/p_A$ & $p_B<y_B$. This contrasts with the tropical dot-bid $(y_A, y_B; x/y_A, x/y_B)$, which is allocated variety A if y_A-$p_A > y_B$-p_B &

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[18] See note 28 of the main text. Baldwin and Klemperer (in prep., b) also show a converse theorem, namely that any constellation of dot-bids that represents preferences that are valid (in the sense described above) represents strong-substitutes preferences.

[19] An equivalent condition is that she views every unit of each variety as mutual substitutes (i.e., if, treating every unit of each variety as a separate product, she views all these products as ordinary substitutes). Strong substitute preferences are also called M-natural, and play an important role in the theory of indivisible goods; they imply the "Law of Aggregate Demand", the "Stepwise Property", the "Consecutive Integer Property", etc., see, e.g., Baldwin and Klemperer (2012). (See also note 7.)

[20] It is well known (see, e.g., Baldwin and Klemperer, 2012, forthcoming) that competitive equilibrium exists when both bidders and the auctioneer express strong-substitutes preferences for indivisible goods. So, if all quantities bid for are integers, our auction (at http://pma.nuff.ox.ac.uk) yields integer quantity allocations to bidders. (However, the (unique lowest) equilibrium price vector may not uniquely define quantity allocations, so an alternative "fairer" rationing of quantities may be preferred if goods are divisible.)

[21] *This must be understood as a conjecture until we have checked the proof.* We also conjecture the converse theorem, namely that any constellation of dot-bids that represents preferences that are valid (in the sense defined above) represents some substitutes preferences.

[22] With agents with (any) substitutes preferences for divisible goods, a (unique) lowest competitive-equilibrium price vector exists, so the Product-Mix Auction price is always well-defined. Baldwin and Klemperer (2012, forthcoming) provide results about when, if all quantities bid for are integers, there exists an equilibrium in which all quantities allocated to bidders are integers (such an equilibrium is a valid solution if bidders regard goods as indivisible, as well as if bidders regard goods as divisible). For more general quasi-linear preferences (including complements), Baldwin and Klemperer provide some results about equilibrium existence, but there may be no unique lowest equilibrium price vector, even when equilibrium exists.

$p_A < y_A$, and variety B if y_B-p_B > y_A-p_A & $p_B < y_B$, although that tropical bid asks for (up to) the same quantities of the varieties, and at up to the same prices. [23]

Fig. 6a illustrates the preferences expressed by a bidder who makes the arctic dot-bid (60, 50; 500): when both prices are below her reservation prices (60,50), she either spends her entire budget on A (and receives q_A=(500/p_A)) or spends her entire budget on B (and receives q_B=(500/p_B)). She is indifferent between these alternatives when (p_B/p_A)=50/60. So the sloping line which divides the region in which she buys only A from the region in which she buys only B passes through the origin, and its slope is the ratio of the elements of the price vector, 50/60. Fig. 6b shows the preferences expressed by a bidder who makes both of the arctic dot-bids (60, 50; 500) and (90, 25; 300).
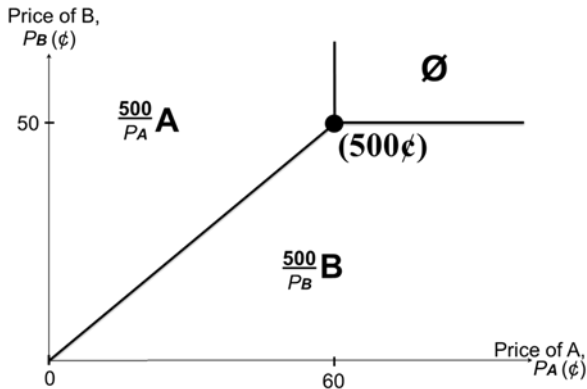

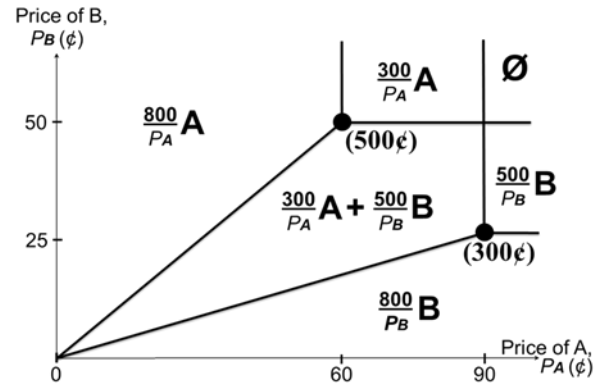Fig. 6a: the Arctic Bid (60, 50; 500) and the preferences it expresses

Fig. 6b: the bid of fig. 6a, together with the Arctic Bid (90, 25; 300), and the preferences expressed by this combination of bids

Fig. 6: Examples of the Arctic Language

Trivially, any bidder with a fixed maximum budget, whose per-unit utility for any given variety is constant (but can depend on the variety), can perfectly represent her preferences using a single positive arctic dot-bid. If the bidder has decreasing marginal utility for some varieties (and so prefers a portfolio of multiple different assets), she can approximate her preferences by breaking her budget up across multiple separate bids. As in the tropical case, making more bids allows a bidder to approximate more complex forms of preferences.

As usual, a merit of our representation of bids in price space is that aggregate demand can be found simply by superimposing the geometric representation of all the individual bidders' bids;[24] finding the outcome of the auction does not require knowing who has made which bid.[25]

C. Extending the languages with new "words"

Especially with more than 2 varieties, some natural aspects of a bidder's preferences may be cumbersome to express with dot-bids. For example, a bidder who wants to select up to *two* pieces of fruit among an apple, a banana, and a clementine, but at most one piece of any variety, needs to use four (tropical) dot-bids (three positive and one negative) to express these preferences.[26] So it can be helpful to define a single

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[23] And when the dot-bids are allocated the same variety, the arctic bid is allocated a greater quantity than its tropical counterpart at any price above the auction price.

[24] In principle, we could allow—and aggregate--both tropical and arctic bids in the same auction.

[25] There were no issues about allocating marginal negative bids (see sec A2a) because we did not permit such bids.

[26] The bids are (V(A), V(B), 0; 1), (V(A), 0, V(C); 1), (0, V(B), V(C); 1), and (V(A), V(B), V(C); -1). (By contrast, the single bid (V(A), V(B), V(C); 2) suffices to demand the two "best-value" items when the same variety can be selected twice.)

special "word" that means "(up to) best two among three varieties" (in which the word states the three specific choices). This both makes expressing preferences simpler, and makes the representation easier to understand.

For example, 5 positive dot-bids plus 4 of these new "words" is more easily understood than the equivalent representation as 5 + (4 x 3) = 17 positive dot-bids plus 4 negative dot-bids. (The latter representation not only contains more vectors; its meaning is also obscured by the entangling of the 16 dot-bids that are best understood as 4 groups of 4 connected dot-bids with each other, and with the 5 dot-bids that are "individuals".)[27]

The (free) software at http://pma.nuff.ox.ac.uk allows the use of "words" such as "best K choices from N varieties".[28],[29]

It is not hard to check that any arctic bid can be approximated arbitrarily closely by a set of tropical bids arrayed along the diagonal line segment from the origin up to the location of that arctic bid. Thus an arctic bid can be understood as another special "word".

D. Allocations from the auctioneer's point of view

We have described the allocations made to a given collection of bids as a function of the price-vector that the auction sets. It is also useful to take the auctioneer's perspective, and show the allocations made to bids as a function of the prices bid, for a given price-vector set by the auction.

Fig. 7a shows the amount allocated to a tropical bid $(y_A, y_B; q_A\ q_B)$ if the *auction* sets the prices (60,50): the bid is allocated q_A of variety A (and no B) if y_A>60 and y_A-60 > y_B-50, is allocated q_B of variety B (and no A) if y_B>50 and y_B-50 > y_A-60, and is allocated nothing if y_A<60 and y_B<50.
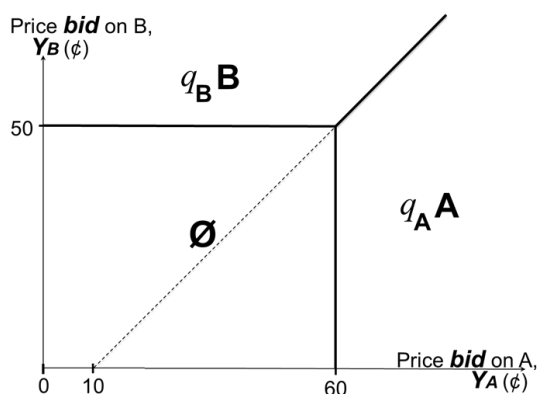

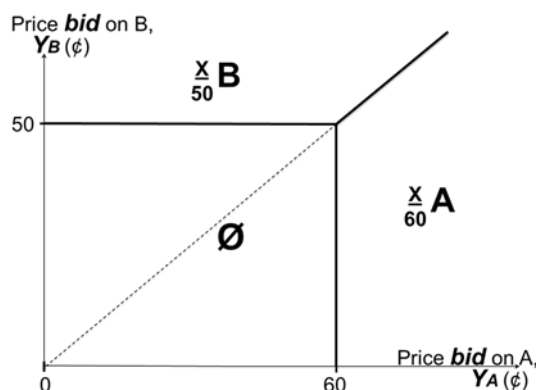Fig. 7a: Allocation to a Tropical Bid, as a function of the *bid's* prices, if the auction sets prices (60,50)

Fig. 7b: Allocation to an Arctic Bid (with budget, x), as a function of the *bid's* prices, if the auction sets prices (60,50)

Fig. 7: Allocations from the Auctioneer's Point of View

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[27] The "best two among three" word helps express preferences such as the demand for "best value from up to 3 apples *and* up to 4 bananas *and* up to 5 clementines *but* not more than 8 pieces of fruit in all". This demand is achieved by the three bids: a bid for 1 clementine (only), a bid for 1 piece of fruit which can be either a banana or a clementine, and a bid for 3 of the "best two among three" of the three varieties (which last bid selects up to 3x2=6 pieces of fruit).

[28] As the number of varieties, N, increases, new words such as these become increasingly beneficial. (In particular, the sets of dot-bids required to pick out K choices from N (with 1<K<N) becomes increasingly complex.)

[29] These "words" are not in the web-app, but are in the command-line version (see the "generalised bids" option).

Fig. 1a (of the main text) illustrates the allocation of multiple tropical bids for an example that was used for testing the Bank of England's original implementation.

Fig. 7b shows the allocation to an arctic bid (y_A, y_B; x) if the auction sets the prices (60,50): this bid is allocated quantity x/60 of variety A (and no B) if y_A>60 and y_A/60 > y_B/50, is allocated is x/50 of variety B (and no A) if y_B>50 and y_B/50 > y_A/60, and is allocated nothing if y_A<60 and y_B<50.

Thus figs. 7a and 7b are "inverses" of figs. 3a and 6a (as is also clear from the discussion of the latter figures in secs. A1a and B, respectively).

E. Expressing the Auctioneer's Preferences

E1) the Bank of England's Original Auctioneer's "Language"

The dot-bid language is not necessarily the most convenient representation for auction participants.

Although the original Bank of England language described in section A1a, above, is a natural one in which to represent an individual bidder's preferences, the Bank of England itself chooses to represent its preferences differently. As described in the main text, for its initial two-variety implementation of the Product-Mix Auction, the auctioneer expressed its preferences as a "supply function" in which the division of the good between the two varieties (the two kinds of loans offered) would depend upon the difference between their prices. It also chose to allocate a fixed total quantity, $Q_A+Q_B=Q$, of the good, subject only to enough bids meeting or exceeding the reserve prices on the respective varieties.

A simple example of such preferences is illustrated in fig. 8a: the auctioneer plans to allocate up to Q=9 units in total. She will sell only B if p_A-p_B < 1 and will sell only A if p_A-p_B > 6; conditional on the reserve prices being met or exceeded, she will sell 3 units of A and 6 units of B if 1< p_A-p_B <4, and will sell 8 units of A and 1 unit of B if 4 < p_A-p_B < 6. So fig. 8a shows the total quantity of A, Q_A, that the auctioneer will
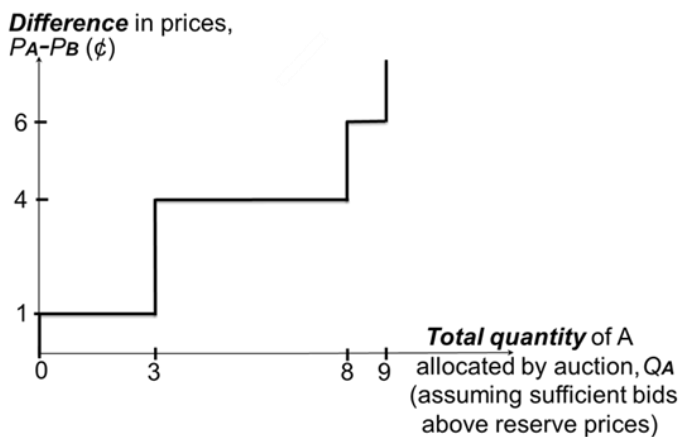

Fig. 8a: Illustration of the Bank of England's supply function, in the original implementation of its auction.
(This is *not* an actual example used by the Bank.)
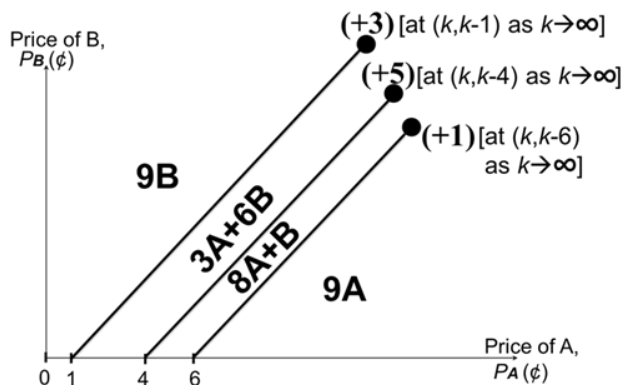
Fig. 8b: Representation in the (ordinary) Bank of England bidding language of the *same* preferences as shown in fig 8a
(Regions of price space are labelled with the total quantities allocated by the auction in those regions = 9A+9B *minus* the demand created by auctioneer's dot-bids)

Fig. 8: Two alternative representations of the *same* preferences of an auctioneer
(illustrating the form of auctioneer's preferences used in the Bank of England's original implementation)

allocate as a function of the difference between the prices, and we can then, of course, also deduce Q_B (=9-Q_A) (assuming the reserve prices are met or exceeded).

Another example is the supply curve shown in fig. 2 of the main text, in which Q_A is given as a percentage of Q (since Q is fixed subject to there being sufficient bids at or above the reserves).

In fact, *any* preferences of this kind can *also* be represented using the simplest "Bank of England" dot-bid language described above—this is not surprising, since the main text shows that the aggregation of all bidder's bids can be represented as a (relative) "demand curve" that has the form of the mirror image of a supply function (see the main text's discussion (in sec. 2) of how we convert the information in fig. 1a into the "demand curve" of fig. 2, via fig. 1b).

For example, the preferences of fig. 8a, can equivalently be represented by thinking of the auctioneer as selling Q=9 units of A *and also* selling Q=9 units of B at any price, and then acting as a bidder who has the preferences shown in fig. 8b and competes with the "ordinary bidders", to buy back at least Q=9 units. In the fig., *k* is chosen large enough that the Q=3 (arbitrarily) high dot-bids (*k*, *k*-1; 3), (*k*, *k*-4; 5) and (*k*, *k*-6; 1), are all always accepted (that is, "bought back" by the auctioneer) on one variety or the other, so exactly Q=9 units are sold (provided a sufficient number of "ordinary bidders' " bids meet the reserve prices[30]). How much of each variety is "bought back" (and so how much of each variety is actually sold to the "ordinary bidders") is determined by how the high dot-bids are allocated, which is precisely determined by the price difference between varieties that the auction sets. (If there are *n* varieties, we think of the auctioneer as selling Q units of each variety, and buying back (*n*-1)Q units at arbitrarily high prices.)

Since the Bank of England preferred to use an approximately continuously increasing supply function, representing it with dot-bids, as in fig. 8b, would be more cumbersome, and less intuitive, than the representation in fig. 8a.

E2) the Bank of England's Current Auctioneer's "Language"

The auctioneer may wish to express richer preferences than the bidders.[31] In particular, the Bank of England's 2014 revision of its Product-Mix Auction extended the auctioneer's "language":

Most important, the Bank of England no longer predetermines the total quantity (summed across all varieties) to allocate, as it did in its original implementation. Instead, the total quantity allocated depends upon the bidding, and is an increasing function of the prices, $p_A, p_B, p_C, \dots$, of the individual varieties. The Bank of England's current implementation permits it to specify some complements preferences, but in a form which retains the uniqueness of the lowest competitive-equilibrium price vector (note that the repos that the Bank of England auctions are divisible).

Second, the Bank of England now auctions more varieties. So, while the original two-variety implementation used a single diagram specifying Q_A/Q as a function of p_A-p_B, in which $Q=Q_A+Q_B$ (as in figs 2 and 8a), their three-variety case uses this diagram (in which now $Q=Q_A+Q_B+Q_C$) *plus* an additional

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[30] One way to implement reserve prices in a Product-Mix Auction is for the auctioneer to make a bid of her own, at the reserve prices, and for a quantity equal to the maximum total quantity available in the auction. (Such a bid will prevent any "ordinary bidder's" bids from winning anything below the reserve price.) For example, inserting a dot-bid (5, 1; 9) in the auction illustrated in figs. 8a and 8b would implement reserve prices of 5 and 1, on varieties A and B respectively. (Ties between "ordinary bidders" and the auctioneer must then be broken in favour of the "ordinary bidders".)

[31] In principle, this seems more likely than the converse, since an individual bidder is interested in the price she pays for, and the composition of, the bundle of varieties she buys herself, while the auctioneer is interested in the revenue from, and the composition of, the entire volume traded. (However, a simple simultaneous multiple round auction allows each individual bidder to express more sophisticated preferences than the auctioneer.)

diagram that specifies $(Q_A+Q_B)/Q$ as a function of $p_B\text{-}p_C$. (With a fourth variety, there would be a third diagram specifying $(Q_A+Q_B+Q_C)/Q$ as a function of $p_B\text{-}p_C$, and $Q=Q_A+Q_B+Q_C+Q_D$, etc.) Preferences of this form are natural when variety A is unambiguously preferred to variety B, which is unambiguously preferred to variety C, …, etc.[32]

Moreover, since the total quantity, Q, is no longer pre-determined, there are additional diagram(s) to determine this.

Precise details of how the Bank of England now specifies its supply preferences are confidential; see also Frost, Govier, and Horn (2015).

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[32] The Bank of England's preferences have this form because the varieties correspond to the minimum qualities of collateral that bidders are required to provide. (The Bank of England labels collaterals in the opposite order; I have renamed them to clarify the relationship between the two- and three- variety case.)

Appendix II (*added 2018*): Iceland's, and Related, Applications

A. Iceland's Planned Implementation

The Icelandic government's strategy for exiting capital controls involved persuading the owners of blocked "offshore" accounts to exchange them for alternative financial assets—either cash, or bonds, with various possible durations and terms, and denominated in either Icelandic krona (ISK) or Euros.[1] Since the government had limited cash, and limited ability to issue bonds, it announced in June 2015 that it would run a product-mix auction in which it, and the owners of the blocked accounts, could express preferences between the varieties of assets for which the blocked funds would be exchanged. The government asked me for (*pro bono*) advice on an appropriate language for its auction, and hired a consulting firm to program it.[2]

Since the owner of any blocked account had a fixed amount of funds available, the Icelandic auction involved "budget-constrained" bidders. Moreover, since each bidder wanted to obtain the best possible portfolio of bonds and cash, given the "budget", x, available to it, bidders cared about price *ratios*: a bidder who values asset J at v_J per unit, would obtain utility $(v_J - p_J).(x/p_J)$ by exchanging her blocked account for asset J, so prefers to exchange her blocked account for A rather than for B if $(p_A/p_B) < (v_A/v_B)$.

So in the bidding language that I proposed, a bid $(y_A, y_B; x)$ is a bid to spend a given *quantity of money*, x, at prices up to (y_A, y_B), and to be allocated variety A if $y_A/p_A > y_B/p_B$ & $p_A < y_A$, and variety B if $y_B/p_B > y_A/p_A$ & $p_B < y_B$. Appendix IB-D gives details.

There are other differences between Iceland's planned implementation of the product-mix auction and the Bank of England's:

The Bank of England pre-specifies its supply functions (although without revealing them to the bidders), but Iceland wished to retain flexibility about the choice of prices until after seeing the bids (in part because of the complexity of its problem). So Iceland's algorithm generated an initial series of options, and then helped the government search across multi-dimensional space for the quantities of different bonds, and Euros, to allocate, and the consequent fraction of ISKs withdrawn. (As usual, simple geometric analysis and pictures were helpful.)

In Iceland's auction, therefore, the allocation across bidders would be fully efficient, but the government had the option to exert some monopsonistic power if it wished. The Bank of England could, if it wished, pre-specify "preferences" so that the auction solution reflects its monopsonistic power, but it chooses not to. Iceland's objectives are confidential.

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[1] The government's concern was that simply unblocking the accounts would be likely to create exchange rate shocks (see Danielsson and Kristjánsdóttir, 2015).

[2] Although the Icelandic government paid a consultancy to program its version of the Product-Mix Auction, the plan to use it was dropped in the 2016 political crisis. The Icelandic government announced (on June 8, 2015) that it would hold a Product-Mix Auction to buy back blocked "offshore" Icelandic funds; it later said it would publish detailed rules in April 2016. But a political crisis erupted on April 3, 2016, when the leaked "Panama papers" revealed that the Prime Minister's wife held over 500 million Icelandic crowns (over $4 million) offshore; the Prime Minister resigned, and there were early elections. There is no reason to believe that a Product-Mix Auction would have benefited the Prime Minister or his wife or anyone else named in the Panama papers; nor, to my knowledge, was this ever suggested. However, the Icelandic Product-Mix Auction was never held. The "offshore" accounts have now (mostly) been unblocked using a more simple-minded approach.

B. Related Problems with "Budget-Constrained" Bidders

Iceland's Product-Mix Auction for budget-constrained bidders has potentially wide application in both public and private-sector contexts.

Sovereign debts could be restructured more efficiently and more cheaply using Iceland's auction. For example, Mexico's "Brady Bond" restructuring allowed the country's creditors to choose to exchange their existing claims for any combination of three alternatives: "discount bonds", "par bonds", and "new money". Moreover, the terms and the relative pricing of the three options were pre-set.[3] Mexico could instead have used Iceland's variant of the Product-Mix Auction to adjust the relative prices and quantities of the three options according to its own and creditors' expressed preferences.

Similarly, shareholders in a firm that is being acquired or otherwise restructured may be offered a choice of alternative securities for their shares. In this case, constraints are typically imposed on the total amount of each option available, so some shareholders will typically be rationed and receive, in part, a different form of consideration than the form they chose.[4] So it would generally be more efficient (and cheaper for an acquiring firm) to use a Product-Mix Auction to set market-clearing prices for the "budget-constrained" shareholders.[5]

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[3] For example, discount bonds were exchanged against existing debt at a 35% discount, par bonds were exchanged at par but had a fixed interest rate of 6.25%, and both these options would receive additional payments based on Mexico's future oil price, etc.; see Claessens and van Wijnbergen (1993).

[4] For example, Walt Disney Company's July 2018 agreement to purchase 21st Century Fox's entertainment assets for (the equivalent of) over $70 billion offered Fox's shareholders the choice of cash or Disney stock subject to a "50% stock, 50% cash proration"; page 4 of the U.S. Securities and Exchange Commission filing at https://www.sec.gov/Archives/edgar/data/1001039/000095015718000748/sc13da.htm explains the rationing (proration), so in this case rationing will almost certainly occur. (Shareholders preferences often differ because of their different tax positions.)

[5]Another problem with a similar flavour is that of students who are given budgets of "points" with which to bid for courses. (See, e.g., Budish, 2011.) However, this problem is a little different because the students' budget constraints are sharper (since "points" are useless after a course-allocation, but blocked Icelandic krona retain some value), and because indivisibilities are crucial (students can't usually take fractional courses).

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